

Resolution of unity for fermionic Gaussian operators

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The fermionic Gaussian operator basis provides a representation for treating strongly correlated fermion systems, as well as playing an important role in random matrix theory. We prove that a resolution of unity exists for any even distribution of eigenvalues over hermitian fermionic Gaussian operators in the nonstandard symmetry classes. This has some important consequences. It demonstrates a useful technique for constructing fundamental mathematical identities in an exponentially complex Hilbert space. It also shows that, to obtain nontrivial results for random matrix canonical ensembles in the nonstandard symmetry classes, it is necessary to consider ensembles that are not even functions of the eigenvalues. We show that the same restriction does not apply to the standard Wigner-Dyson symmetry classes of random matrices.

I. INTRODUCTION

The linear transformations of fermion operators that preserve anti-commutation relations is an important fundamental symmetry group. It is known to have a representation as an exponential of a general quadratic form in the fermion operators. By analogy with the Gaussian function in statistics, we term this a fermionic Gaussian operator. The group properties and integration measures of these operators have been extensively investigated in the mathematical physics literature^{1–6}. They have been used to generate the generalized fermionic coherent states^{7–9}. This group is also known to correspond to a general symmetry class in random matrix theory^{2,10,11}.

The Gaussian fermion operators have other fundamental applications. They are an operator basis for all fermionic density operators. Both the well-known thermal states and the BCS states are examples of this. The Gaussian fermionic phase-space representation^{12–14} makes use of this property to generate a positive distribution over such operators. This is a complete phase-space representation for either fermionic and bosonic many body density operators. It has been used, for example, to computationally evaluate the ground state of the fermionic Hubbard model¹⁵ and to obtain an expression for the linear entropy of a density operator¹⁶.

In this communication, we show that an expansion using hermitian fermionic Gaussian operators provides a continuous positive-definite resolution of the fermionic identity operator. Our result holds for any integrable even distribution of eigenvalues. This resolution is complete, in that it generates an identity operator for the entire Hilbert space, not a subset. The proof of the resolution of unity is given in terms of the polar coordinates of skew symmetric matrices. This shows that the fermionic Gaussian operators can be thought of as coherent operators. In terms of continuity and completeness, they have an analogous role to the bosonic coherent state projectors^{17,18}.

To understand this comparison, we recall that one of the most useful properties of the bosonic coherent states is their resolution of unity^{17,19,20}. This resolution of the bosonic identity operator uses positive-definite coherent state projection operators, with a positive measure over complex amplitudes. It is the basis for such well-known methods as the Husimi Q-function²¹. Our resolution of unity for the fermionic Gaussian operators has similar properties, with a normalized Gaussian operator replacing the coherent-state projector in the expansion of the unity. Just as in the boson case, a resolution of unity is the foundation for calculating other fermionic identities and phase-space representations. These will be treated in subsequent publications.

An alternative approach by Gilmore^{22,23} and Perelomov^{24,25} defined coherent state representa-

tions for fermions using generalized fermionic coherent states. These are obtained as the action of a representation of a Lie group on an extremal state. This definition is related to the dynamical structure of the corresponding creation and annihilation operators, using $U(N)$ Lie group methods^{7–9,18,26–30}. In this approach, an integral over the coherent state projectors appears, but it does *not* generate an identity operator for all fermion states, due to number conservation. Because the fermionic coherent states are defined only for a subset of the Hilbert space, it is not possible to obtain a continuous representation of unity.

Phase space representations for fermions³¹ can also be defined in terms of Grassmann coherent states^{27,31–33}, using anti-commuting Grassmann variables³⁴. As the Grassmann coherent states form a complete set, it is possible in this case to obtain a resolution of unity. However, the use of Grassmann variables means that the phase-space is defined over non-commuting variables. These have an exponentially large matrix representation in terms of standard complex variables. Consequently, they have no efficient direct computational representation without requiring exponentially large computational resources. In addition, while Grassmann coherent states are mathematically well-defined, they do not correspond to any physical state.

There is a close relationship between our work and the theory of random matrices. The transformations we use correspond to the nonstandard symmetry classes of random matrices. In this context, an important consequence of our result is that it provides an exact solution to a random matrix distribution of fermionic canonical ensembles. We show that in the nonstandard symmetry classes, an even distribution over matrix eigenvalues reduces a general canonical ensemble to the identity operator. In other words, canonical ensembles in these symmetry classes are nontrivial only if the random matrix distributions are *not* even functions of the eigenvalue. Nonstandard classes of transformations can still have useful properties if the eigenvalue distributions are not even functions - for example, if there is a lower cutoff to the eigenvalue distribution. Similarly, our result clearly does not exclude the use of ensembles that are not canonical.

This paper is organized as follows: In section II we define the general Gaussian operators. In section III, we define the symmetry classes that we use. In section IV we give the resolution of unity for the general Gaussian operators, and in section V we consider the case of the number-conserving Gaussian operators. Section VI gives a summary of our results, conclusions and outlook.

Additional results and integration identities are given in the Appendix.

II. FERMIONIC GAUSSIAN OPERATORS

We start by reviewing the known properties of fermionic Gaussian operators. This will help to establish the notation, as well as to give results that will be used in later parts of the paper. We consider a fermionic system that can be decomposed into M spatial or internal modes. In some treatments of symmetry properties the space and spin indices are treated separately, but this is not essential to our results. We define $\hat{\mathbf{a}}$ as a vector of M annihilation operators and $\hat{\mathbf{a}}^\dagger$ as vector of M creation operators. As they are Fermi operators, \hat{a}_i and \hat{a}_j^\dagger obey the anticommutation relations:

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}, \quad \{\hat{a}_i, \hat{a}_j\} = 0. \quad (2.1)$$

We define an extended vector of all $2M$ operators $\hat{\boldsymbol{\gamma}} = (\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger)$, with an adjoint vector defined as $\hat{\boldsymbol{\gamma}}^\dagger = (\hat{\mathbf{a}}^\dagger, \hat{\mathbf{a}}) = (\hat{a}_1^\dagger, \dots, \hat{a}_M^\dagger, \hat{a}_1, \dots, \hat{a}_M)$. In the remainder of this section, we will summarize properties of the fermionic Gaussian operators to be used later.

A. Group properties

Following the original work of Balian and Brezin¹, we consider general linear transformations of fermions. These are obtained from the fermionic Gaussian operator defined as an exponential of a general quadratic form in Fermi operators:

$$\hat{G}_B(\mathbf{R}) = \exp \left[\frac{1}{2} \boldsymbol{\gamma} \mathbf{R} \boldsymbol{\gamma} \right], \quad (2.2)$$

where \mathbf{R} is a $2M \times 2M$ antisymmetric complex matrix. The group composition law for the general Gaussian operator is

$$\hat{G}_B(\mathbf{R}) = \hat{G}_B(\mathbf{R}^{(1)}) \hat{G}_B(\mathbf{R}^{(2)}), \quad (2.3)$$

The value of \mathbf{R} under this composition law is obtained on defining:

$$\mathbf{H} = \boldsymbol{\sigma} \mathbf{R}, \quad (2.4)$$

where we have introduced a symmetric matrix:

$$\boldsymbol{\sigma} = \begin{pmatrix} \mathbf{0}_M & \mathbf{I}_M \\ \mathbf{I}_M & \mathbf{0}_M \end{pmatrix}. \quad (2.5)$$

With this definition, the matrices follow a composition law for the matrix parameters given by:

$$\exp(\mathbf{H}) = \exp(\mathbf{H}^{(1)}) \exp(\mathbf{H}^{(2)}). \quad (2.6)$$

This is clearly a Lie group. From the composition law, it has an inverse and an identity, as well as being associative and differentiable. It is equivalent to the $2M \times 2M$ complex orthogonal Lie group, which follows¹ from the anti-symmetry of \mathbf{R} .

B. Hermitian sub-group

The general Gaussian operator for fermions can also be usefully defined in terms of the \mathbf{H} matrix. Since this is the form most commonly used in later work, we will use it here. With this definition, the general un-normalized fermionic Gaussian operator is:

$$\hat{G}(\mathbf{H}) = \exp \left[\frac{1}{2} \hat{\boldsymbol{\gamma}}^\dagger \mathbf{H} \hat{\boldsymbol{\gamma}} \right]. \quad (2.7)$$

Here \mathbf{H} is an $2M \times 2M$ matrix, whose definition in terms of an antisymmetric matrix \mathbf{R} implies that it has the decomposition

$$\mathbf{H} = \begin{pmatrix} \mathbf{h} & \Delta \\ -\Delta^* & -\mathbf{h}^T \end{pmatrix}, \quad (2.8)$$

where \mathbf{h} and Δ are $M \times M$ matrices. Hence the quadratic term in the exponent can be rewritten as:

$$\hat{H} = \frac{1}{2} \hat{\boldsymbol{\gamma}}^\dagger \mathbf{H} \hat{\boldsymbol{\gamma}} = \frac{1}{2} \left(\hat{\mathbf{a}}^\dagger \mathbf{h} \hat{\mathbf{a}} - \hat{\mathbf{a}} \mathbf{h}^T \hat{\mathbf{a}}^\dagger + \hat{\mathbf{a}}^\dagger \Delta \hat{\mathbf{a}}^\dagger - \hat{\mathbf{a}} \Delta^* \hat{\mathbf{a}} \right). \quad (2.9)$$

We note that, as well as being useful for linear transformations, the same class of fermionic Gaussian operators can also be used to expand a general fermionic density matrix in a positive phase-space distribution¹²⁻¹⁴. In general, this distribution includes both hermitian and non-hermitian operators.

In this form it is clear that if we wish to restrict the Gaussian operators to be hermitian, we must require that $\mathbf{h} = \mathbf{h}^\dagger$, and $\Delta = -\Delta^T$. This is the form we will use here to establish the resolution of the fermionic identity operator. With this hermitian restriction, the quadratic form has a clear physical identification. It is simply the Bogoliubov-de Gennes Hamiltonian obtained from linearizing the Hamiltonian for a superconductor. From this perspective, we see that the Gaussian operator can have intrinsic coherence properties that correspond to either a superconductor with $\Delta \neq 0$, or to a normal fluid with $\Delta = 0$. Because it carries phase information, the matrix Δ plays a similar role in fermion physics to the bosonic coherent state amplitude, which appears in various approaches to laser and superfluid theory.

C. Number-conserving case

Next we consider a subset of the general Gaussian operators with $\Delta = 0$. These are the *number-conserving* fermionic Gaussian operators. In this case, $\hat{\mathbf{H}}$ corresponds to the fermionic Hamiltonian of a non-interacting Fermi gas in an arbitrary spin-dependent potential. In general, these Gaussian operators form a subgroup of the complex linear group, $GL(M, \mathbb{C})$. Balian and Brezin¹ found that these operators can be rewritten in terms of a single $M \times M$ matrix:

$$\hat{G}_N(\mathbf{h}) = \exp \left[\hat{\mathbf{a}}^\dagger \mathbf{h} \hat{\mathbf{a}} - \frac{1}{2} \text{Tr}(\mathbf{h}) \right]. \quad (2.10)$$

Using the matrix identity $\exp(\text{Tr}(\mathbf{h})) = \det(\exp(\mathbf{h}))$, the number-conserving Gaussian operators can be written as:

$$\hat{G}_N(\mathbf{h}) = \frac{1}{\sqrt{\det(\exp(\mathbf{h}))}} \exp \left[\hat{\mathbf{a}}^\dagger \mathbf{h} \hat{\mathbf{a}} \right]. \quad (2.11)$$

The group composition law for these operators is given by:

$$\hat{G}_N(\mathbf{h}) = \hat{G}_N(\mathbf{h}^{(1)}) \hat{G}_N(\mathbf{h}^{(2)}), \quad (2.12)$$

Here the composition law for \mathbf{h} is obtained on matrix multiplication of the variables $\mathbf{u} = e^{\mathbf{h}}$. The reason for this is that multiplying the $\mathbf{u}^{(j)}$ matrices is equivalent to multiplying the $\exp(\mathbf{H}^{(j)})$ matrices defined in Eq. (2.6). Hence, we see that:

$$e^{\mathbf{h}} = e^{\mathbf{h}^{(1)}} e^{\mathbf{h}^{(2)}}. \quad (2.13)$$

We can equivalently express this Gaussian operator in terms of normally-ordered parameters, using the following general mathematical identity for an M -mode fermionic operator³⁵:

$$\exp \left[\hat{\mathbf{a}}^\dagger \mathbf{h} \hat{\mathbf{a}} \right] = : \exp \left[\hat{\mathbf{a}}^\dagger \left[e^{\mathbf{h}} - \mathbf{I} \right] \hat{\mathbf{a}} \right] :. \quad (2.14)$$

Therefore we obtain:

$$\hat{G}_N(\mathbf{h}) = e^{-\frac{1}{2} \text{Tr}[\mathbf{h}]} : \exp \left[\hat{\mathbf{a}}^\dagger \left[e^{\mathbf{h}} - \mathbf{I} \right] \hat{\mathbf{a}} \right] :. \quad (2.15)$$

Hence, we notice that if we use normal ordering the simplest parameterization is through the parameter $\mathbf{u} = e^{\mathbf{h}}$, which gives a matrix-multiplication group composition law for these operators. The quadratic term in the exponent can also be written in a Hamiltonian-like form, in terms of a thermal or number-conserving Hamiltonian $\hat{\mathbf{H}}_N$:

$$\hat{H}_N = \hat{\mathbf{a}}^\dagger \mathbf{h} \hat{\mathbf{a}} - \frac{1}{2} \text{Tr}(\mathbf{h}) = \frac{1}{2} \left(\hat{\mathbf{a}}^\dagger \mathbf{h} \hat{\mathbf{a}} - \hat{\mathbf{a}} \mathbf{h}^T \hat{\mathbf{a}}^\dagger \right). \quad (2.16)$$

We recognize that to obtain a hermitian Gaussian operator, it is necessary to restrict \mathbf{h} to the class of hermitian matrices.

III. SYMMETRY CLASSES AND GROUPS

The study of general symmetry groups for many-body systems originates in the work of Wigner^{36,37} and Dyson^{3,4}, who studied the energy levels of complex many body systems such as nuclei. In particular, Dyson^{3,4} classified many-body systems depending on the symmetry properties of the Hamiltonian. He classified Hamiltonians according to their time-reversal and rotational invariance properties.

In random matrix theory, this classification corresponds to three random matrix models: the Gaussian unitary ensemble (GUE), the Gaussian orthogonal ensemble (GOE), and the Gaussian symplectic ensemble (GSE)^{11,38}. The unitary ensemble applies to general systems without any invariance under time reversal, the orthogonal ensemble is for systems with time-reversal and rotational invariance, while the symplectic ensemble is for systems with time-reversal invariance but no rotational invariance^{11,38}.

Dyson's classification is based on number-conserving Hamiltonians. If we consider the more general class of non-number-conserving Hamiltonians, there are four additional symmetry classes that can be identified depending on their time reversal symmetry and spin-rotation invariance². These four symmetry classes are Class D, Class C, Class DIII and Class CI, which are related to Cartan's classification of symmetric spaces. Class D corresponds to systems with neither time-reversal symmetry nor spin-rotation symmetry. Class DIII corresponds to systems with time-reversal symmetry but no spin-rotation invariance. Class C is for systems with spin-rotation invariance and no time-reversal invariance, while Class CI is for systems with spin-rotation invariance and time-reversal invariance.

A. Gaussian operators in nonstandard symmetry classes

The nonstandard symmetry classes are the general symmetry classes of matrices that involve non-number-conserving Hamiltonians and particle-hole symmetry. Here we focus mainly on the Class D symmetry defined by Altland and Zirnbauer², although our results are applicable to any of the nonstandard symmetry classes. Class D corresponds to the most general nonstandard symmetry class, as it applies to cases which do not have time-reversal or spin-rotation symmetry. In this case the only required properties of the matrices \mathbf{h} and $\mathbf{\Delta}$, defined in Eq. (2.8), are the hermiticity of \mathbf{h} and skew-symmetry of $\mathbf{\Delta}$. The $2M \times 2M$ \mathbf{H} matrix is therefore hermitian, as discussed above.

We start by summarizing group theoretic results of Altland and Zirnbauer, then extend these to obtain new results for distributions over canonical ensembles.

Hermitian matrices do not themselves form a group because they do not close under commutation. However, the anti-hermitian matrices form a Lie group, so it is useful to define an anti-hermitian matrix \mathbf{X} , as $\mathbf{X} = i\mathbf{H}$. Hence the conditions on \mathbf{h} and $\mathbf{\Delta}$ can be expressed in terms of the anti-hermitian matrix \mathbf{X} as:

$$-\mathbf{X}^\dagger = \mathbf{X} = -\mathbf{\Sigma}_X \mathbf{X}^T \mathbf{\Sigma}_X, \quad (3.1)$$

where the matrix $\mathbf{\Sigma}_X$ is defined as:

$$\mathbf{\Sigma}_X = \begin{pmatrix} \mathbf{0}_M & \mathbf{I}_M \\ \mathbf{I}_M & \mathbf{0}_M \end{pmatrix}.$$

The Lie algebra of the matrices defined in Eq. (3.1) is isomorphic to the $so(2M)$ algebra². Since \mathbf{X} belongs to a Lie algebra, it can be diagonalized:

$$\mathbf{X} = \mathbf{U}^{-1} \tilde{\boldsymbol{\lambda}} \mathbf{U}. \quad (3.2)$$

Here \mathbf{U} is an element of a Lie group which is isomorphic to $SO(2M)$. The diagonalization of Eq. (3.2) is equivalent to decomposing the \mathbf{X} matrix in matrix polar coordinates. The radial part corresponds to the $\tilde{\boldsymbol{\lambda}}$ matrix, $\tilde{\boldsymbol{\lambda}} = \text{diag}(i\boldsymbol{\lambda}, -i\boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ is an $M \times M$ diagonal matrix of eigenvalues $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_M\}$, and the angular part corresponds to the \mathbf{U} matrix.

B. Diagonal form

Let us now consider the exponent of the Gaussian operator defined in Eq. (2.7). After diagonalization, we can write this operator expression in terms of the matrix \mathbf{X} as:

$$\begin{aligned} \hat{H} &= \frac{1}{2} \hat{\boldsymbol{\gamma}}^\dagger \mathbf{H} \hat{\boldsymbol{\gamma}} = \frac{1}{2} \hat{\boldsymbol{\gamma}}^\dagger (-i\mathbf{X}) \hat{\boldsymbol{\gamma}} \\ &= \frac{-i}{2} (\hat{\mathbf{a}}^\dagger, \hat{\mathbf{a}}) (\mathbf{U}^{-1} \tilde{\boldsymbol{\lambda}} \mathbf{U}) \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{a}}^\dagger \end{pmatrix}. \end{aligned}$$

Since the canonical anticommutation relations are invariant under this transformation, we can define new fermionic operators

$$\begin{pmatrix} \hat{\mathbf{b}} \\ \hat{\mathbf{b}}^\dagger \end{pmatrix} = \mathbf{U} \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{a}}^\dagger \end{pmatrix}, \quad (3.3)$$

therefore, we can show that:

$$\hat{G}(\mathbf{H}) = \exp \left[\frac{-i}{2} (\hat{\mathbf{b}}^\dagger, \hat{\mathbf{b}}) \begin{pmatrix} i\boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & -i\boldsymbol{\lambda} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{b}} \\ \hat{\mathbf{b}}^\dagger \end{pmatrix} \right],$$

and hence using anti-commutation relations it follows that:

$$\hat{G}(\boldsymbol{\lambda}) = \exp \left[\boldsymbol{\lambda} \hat{\mathbf{b}}^\dagger \hat{\mathbf{b}} - \frac{1}{2} \text{Tr}[\boldsymbol{\lambda}] \right]. \quad (3.4)$$

We can write Eq. (3.4) in normally ordered form using Eq. (2.14), hence:

$$\hat{G}(\boldsymbol{\lambda}) = e^{-\frac{1}{2} \text{Tr}[\boldsymbol{\lambda}]} : \exp \left[\hat{\mathbf{b}}_i^\dagger (e^{\lambda_i} - 1) \hat{\mathbf{b}}_i \right] :. \quad (3.5)$$

C. Elementary properties

We now wish to use this diagonal form to prove some elementary properties of these Gaussian operators.

1. Positivity

The Gaussian operator defined in Eq. (3.5) is defined in terms of real eigenvalues, $\lambda_j \in (-\infty, \infty)$, so that $e^{\lambda_j} \in (0, \infty)$. Hence, if we consider the normally ordered form we notice that

$$\begin{aligned} \hat{G}(\boldsymbol{\lambda}) &= e^{-\frac{1}{2} \text{Tr}[\boldsymbol{\lambda}]} : \exp \left[\hat{\mathbf{b}}_i^\dagger (e^{\lambda_i} - 1) \hat{\mathbf{b}}_i \right] : \\ &= \prod_i e^{-\lambda_i/2} \left(1 + \hat{\mathbf{b}}_i^\dagger (e^{\lambda_i} - 1) \hat{\mathbf{b}}_i \right) \geq 0. \end{aligned} \quad (3.6)$$

Therefore the Gaussian operators with hermitian \mathbf{H} matrices are themselves hermitian, positive definite operators in the fermionic Hilbert space.

2. Normalization

The Gaussian operators defined in Eq. (3.5) can be normalized to obtain operators $\hat{\Lambda}(\mathbf{H})$, such that $\text{Tr}[\hat{\Lambda}(\mathbf{H})] = 1$. We first consider the trace of a single-mode case, which is given by:

$$\text{Tr} \left[: \exp \left[\hat{b}^\dagger (e^\lambda - 1) \hat{b} \right] : \right] = \text{Tr} \left[1 + \hat{b}^\dagger \hat{b} (e^\lambda - 1) \right] = 1 + e^\lambda. \quad (3.7)$$

On the other hand, including the exponential factor obtained during normal-ordering,

$$\text{Tr} \left[e^{\hat{b}^\dagger \lambda \hat{b} - \frac{\lambda}{2}} \right] = \left(1 + e^\lambda \right) e^{-\frac{\lambda}{2}} = 2 \cosh \left(\frac{\lambda}{2} \right). \quad (3.8)$$

Then for the single mode case we can introduce a normalized Gaussian operator $\hat{\Lambda}(\lambda)$ such that $Tr [\hat{\Lambda}(\lambda)] = 1$, in the form $\hat{\Lambda}(\lambda) = \exp \left[\hat{b}^\dagger \lambda \hat{b} - \frac{\lambda}{2} \right] / [2 \cosh (\lambda/2)]$.

For the general M -mode case we can transform back to the original fermionic operators, using invariance of the determinant under unitary transformations to obtain:

$$\hat{\Lambda}(\mathbf{H}) = \frac{e^{\frac{1}{2} \hat{\gamma}^\dagger \mathbf{H} \hat{\gamma}}}{\det [2 \cosh (\mathbf{H}/2)]}. \quad (3.9)$$

We note that this normalized form is identical to that used in Gaussian phase-space representations^{12–14}.

IV. GENERAL RESOLUTION OF UNITY

A. General Gaussian Operators

We wish to prove that the resolution of unity for the normalized hermitian Gaussian operator, $\hat{\Lambda}(\mathbf{H})$, defined in Eq. (3.9) is given by:

$$\int d\mathbf{H} \hat{\Lambda}(\mathbf{H}) P(\mathbf{H}^2) = \hat{I}. \quad (4.1)$$

Here \hat{I} is the fermionic identity operator, $d\mathbf{H}$ is the measure over hermitian matrices which conserves the nonstandard group symmetries of any of the four different symmetry classes defined by Altland and Zirnbauer². Here, $P(\mathbf{H}^2)$ is a normalizable positive function of \mathbf{H}^2 that is invariant under the transformation \mathbf{U} , where \mathbf{U} depends on the symmetry class. On taking a trace of the equation, we note from Eq (4.1), that the normalization of this function must satisfy: $2^{-M} \int d\mathbf{H} P(\mathbf{H}^2) = 1$.

Although our result is general, for definiteness we consider two options for the normalization function:

$$P^{(1)} = C^{(1)} \det [1 + \mathbf{H}^2]^{-p}, \quad (4.2)$$

$$P^{(2)} = C^{(2)} \exp [-p \text{Tr} [\mathbf{H}^2]]. \quad (4.3)$$

B. Matrix polar coordinates

We now wish to use a unitary transformation to reduce the integral over matrix elements of \mathbf{H} to matrix polar coordinates. The angular variables correspond to unitary transformations, while radial variables correspond to eigenvalues. The Jacobian for the transformation from the Cartesian

Table I. Indices of the different symmetry classes.

Class	β	α
D	2	0
C	2	2
DIII	4	1
C1	1	1

coordinates $\mathbf{H} = -i\mathbf{X}$ to polar coordinates $(\boldsymbol{\lambda}, \mathbf{U})$, is given by diagonalizing the matrix \mathbf{H} defined in Eq. (2.8), so that²

$$d\mathbf{H} = (\mathbf{U}^\dagger d\mathbf{U}) \Delta^\beta(\boldsymbol{\lambda}^2) d\boldsymbol{\lambda} \prod_j |\lambda_j|^\alpha. \quad (4.4)$$

Here $d\boldsymbol{\lambda} = \prod_{j=1}^M d\lambda_j$, \mathbf{U} is the transformation that diagonalizes the matrix \mathbf{H} for the symmetry class under consideration, and $\Delta(\boldsymbol{\lambda})$ is the Vandermonde determinant defined as:

$$\Delta(\boldsymbol{\lambda}) = \Delta(\lambda_1, \dots, \lambda_M) = \prod_{1 \leq i < j \leq M} (\lambda_i - \lambda_j) = \det [\lambda_i^{j-1}]. \quad (4.5)$$

The indices α and β of Eq. (4.4), depend on the underlying nonstandard symmetry class. Their values are given in Table I.

Eq. (4.1) in polar coordinates can now be written, after unitary transformation to diagonal operator form, as:

$$\begin{aligned} \int d\mathbf{H} \hat{\Lambda}(\mathbf{H}) P(\mathbf{H}^2) &= \int (\mathbf{U}^\dagger d\mathbf{U}) \int P(\boldsymbol{\lambda}^2) \Delta^\beta(\boldsymbol{\lambda}^2) d\boldsymbol{\lambda} \times \\ &\times \prod_{j=1}^M \left[\frac{|\lambda_j|^\alpha e^{-\frac{1}{2}\lambda_j} : \exp \left[\hat{b}_j^\dagger (e^{\lambda_j} - 1) \right]_j \hat{b}_j}{[2 \cosh(\lambda_j/2)]} \right]. \end{aligned} \quad (4.6)$$

Just as in the previous section, we have defined \hat{b}_j as a function of the unitary transformation \mathbf{U} , according to Eq (3.3).

C. Evaluation of integrals

We next wish to show that the result of the integral over radial variables $\boldsymbol{\lambda}$ is a constant, independent of the transformation \mathbf{U} . In order to evaluate the integral over the radial variables $\boldsymbol{\lambda}$, we

notice that the expansion of the operator term can be expressed as:

$$\begin{aligned} \prod_{j=1}^M e^{-\frac{1}{2}\lambda_j} : \exp \left[\hat{\mathbf{b}}_j^\dagger (e^{\lambda} - 1)_j \hat{\mathbf{b}}_j \right] &:= \prod_{j=1}^M e^{-\frac{1}{2}\lambda_j} \left(1 + \hat{b}_j^\dagger (e^{\lambda} - 1)_j \hat{b}_j \right) \\ &= \prod_{j=1}^M \left(e^{-\frac{1}{2}\lambda_j} + 2 \sinh \left(\frac{\lambda_j}{2} \right) \hat{b}_j^\dagger \hat{b}_j \right). \end{aligned} \quad (4.7)$$

Using this result to expand Eq. (4.6) we obtain:

$$\begin{aligned} \int d\lambda \Delta^\beta(\lambda^2) \hat{\Lambda}(\lambda) P(\lambda^2) \prod_{j=1}^M |\lambda_j|^\alpha &= \int d\lambda P(\lambda^2) \Delta^\beta(\lambda^2) \prod_{j=1}^M \left[\frac{|\lambda_j|^\alpha e^{-\frac{1}{2}\lambda_j}}{[2 \cosh(\lambda_j/2)]} \right] \\ &\quad + \int d\lambda P(\lambda^2) \Delta^\beta(\lambda^2) \prod_{j=1}^M \left[|\lambda_j|^\alpha \tanh \left(\frac{\lambda_j}{2} \right) \hat{b}_j^\dagger \hat{b}_j \right]. \end{aligned} \quad (4.8)$$

Here we see that the integral over the operator terms $\hat{b}_i^\dagger \hat{b}_i$ has tanh terms which are all odd in the λ_j variable, while every other term is an even function of λ_j . Therefore, from the parity of these functions, all the terms of the second integral in Eq. (4.8) vanish on integration over λ_j .

Since the operator terms are the only terms that depend on the unitary transformation \mathbf{U} , it follows that the integral over the angular part is a constant, given by the relevant angular volume, which we will denote by $C^{\mathbf{U}}$. The value of $C^{\mathbf{U}}$ depends on the corresponding non-standard symmetry class under consideration, which defines the transformation \mathbf{U} . The result of the integration over the eigenvalues then determines the normalization factor.

Using the standard identity, $\exp(\lambda_j/2) = \cosh(\lambda_j/2) + \sinh(\lambda_j/2)$, Eq. (4.8) can be written as:

$$\begin{aligned} \int d\mathbf{H} \hat{\Lambda}(\mathbf{H}) P(\mathbf{H}^2) &= C^{\mathbf{U}} 2^{-M} \int d\lambda P(\lambda^2) \Delta^\beta(\lambda^2) \prod_{j=1}^M |\lambda_j|^\alpha \left[1 - \tanh \left(\frac{\lambda_j}{2} \right) \right] \\ &= 2^{-M} C^{\mathbf{U}} \int_{-\infty}^{\infty} d\lambda P(\lambda^2) \Delta^\beta(\lambda^2) \prod_{j=1}^M |\lambda_j|^\alpha. \end{aligned} \quad (4.9)$$

In order to obtain Eq. (4.9), we have used the result that the integral over the $\tanh(\lambda_j/2)$ terms vanishes, because clearly $\tanh(\lambda_j/2)$ is an odd function of λ_j while the other terms are even. Next, we recall that the original definition of P was such that it was normalized. Therefore, using matrix polar coordinates to evaluate Eq (4.1), we find that:

$$\begin{aligned} 1 &= 2^{-M} \int d\mathbf{H} P(\mathbf{H}^2) \\ &= 2^{-M} C^{\mathbf{U}} \int_{-\infty}^{\infty} d\lambda P(\lambda^2) \Delta^\beta(\lambda^2) \prod_{j=1}^M |\lambda_j|^\alpha. \end{aligned} \quad (4.10)$$

For any non-standard symmetry class it is therefore possible to express the resolution of unity as:

$$\begin{aligned}\hat{I} &= \int d\mathbf{H} \hat{\Lambda}(\mathbf{H}) P(\mathbf{H}^2) \\ &= 2^{-M} C^{\mathbf{U}} \int_{-\infty}^{\infty} d\boldsymbol{\lambda} P(\boldsymbol{\lambda}^2) \Delta^{\beta}(\boldsymbol{\lambda}^2) \prod_{j=1}^M |\lambda_j|^{\alpha}.\end{aligned}\quad (4.11)$$

The value of both the angular volume $C^{\mathbf{U}}$ and the radial integral will depend on the corresponding non-standard symmetry class under consideration. For the radial part, the integral will also depend on the different choices of the normalization function $P(\boldsymbol{\lambda}^2)$. We notice that as long as we can perform the integration over the angular and radial part, then it is possible to obtain a resolution of unity for any of the non-standard symmetry classes. An explicit result for these integrals in the case of class D symmetry is given next.

We note there is an important consequence of this result in random matrix theory. Suppose we consider a statistical random matrix mixture of finite temperature canonical ensembles given by:

$$\rho = \int d\mathbf{H} P(\mathbf{H}^2) \exp[-\beta \hat{H}] \propto \hat{I}, \quad (4.12)$$

where \hat{H} is the linearized Bogoliubov-de Gennes Hamiltonian given by Eq (2.9), and $d\mathbf{H}$ is a measure over one of the four nonstandard symmetry classes. It follows that any correlation function or moment evaluated in this ensemble is simply an average over the identity operator, independent of temperature, symmetry class or the details of the distribution $P(\mathbf{H}^2)$.

D. Symmetry Class D

In order to give the values of the angular volume and radial integrals defined in Eq. (4.11), we will consider the most general symmetry class D. That is, we are going to integrate over the transformations described in Sec. III A. We note, however, that similar results in any of the non-standard symmetry classes can be found. The main differences are in the values of the integration volumes, and in convergence properties which depend on the details of the Jacobian and matrix polar coordinates.

For symmetry class D, which is the largest symmetry class, the values of the indices α and β , given in Table I, are 0 and 2 respectively². In this case the value of the integral over the angular

part, $C^{\mathbf{U}} = \mathbf{U}^\dagger d\mathbf{U}$ is given below. Details are given in the Appendix:

$$C^{\mathbf{U}} = \int (\mathbf{U}^\dagger d\mathbf{U}) = \frac{\pi^{M(M-\frac{1}{2})}}{2^{M(M-1)}} \prod_{j=0}^{M-1} \frac{1}{\Gamma(2+j) \Gamma(j+\frac{1}{2})}. \quad (4.13)$$

Hence, for the non-standard symmetry class D, the integral defined in Eq. (4.11) is:

$$\begin{aligned} \hat{I} &= \int d\mathbf{H} \hat{\Lambda}(\mathbf{H}) P(\mathbf{H}^2) \\ &= 2^{-M} C^{\mathbf{U}} \int_{-\infty}^{\infty} P(\boldsymbol{\lambda}^2) \Delta^2(\boldsymbol{\lambda}^2) d\boldsymbol{\lambda}. \end{aligned} \quad (4.14)$$

1. Determinant normalization

In order to evaluate the integral over the radial variable $\boldsymbol{\lambda}$, we have to consider one of the two options for the normalization $P(\boldsymbol{\lambda}^2)$. We first consider the normalization by the determinant given in Eq. (4.2). Hence the integral over the variables $\boldsymbol{\lambda}$ is:

$$\int_{-\infty}^{\infty} d\boldsymbol{\lambda} \Delta^2(\boldsymbol{\lambda}^2) P^{(1)}(\boldsymbol{\lambda}^2) = C^{(1)} \int_{-\infty}^{\infty} d\boldsymbol{\lambda} \Delta^2(\boldsymbol{\lambda}^2) \prod_{j=1}^M (1 + \lambda_j^2)^{-2p}. \quad (4.15)$$

The integral of the right hand side of Eq. (4.15) is known as Selberg's Integral³⁸. Selberg's formula is given below. It is valid for integer n and complex α, β, γ with $\text{Re}\alpha > 0$, $\text{Re}\beta > 0$ and $\text{Re}\gamma > -\min(\frac{1}{n}, \frac{\text{Re}\alpha}{n-1}, \frac{\text{Re}\beta}{n-1})$:

$$\begin{aligned} I(\alpha, \beta, \gamma, n) &= \int_0^\infty \dots \int_0^\infty |\Delta(\mathbf{x})|^{2\gamma} \prod_{j=1}^n x_j^{\alpha-1} (1+x_j)^{-\alpha-\beta-2\gamma(n-1)} dx_j \\ &= \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma) \Gamma(\alpha+j\gamma) \Gamma(\beta+j\gamma)}{\Gamma(1+\gamma) \Gamma(\alpha+\beta+(n+j-1)\gamma)}. \end{aligned} \quad (4.16)$$

Defining $v_j = \lambda_j^2$, $dv = 2\lambda d\lambda_j$, $d\lambda_j = \frac{1}{2} v^{-\frac{1}{2}} dv$, and setting $n = M$, $\alpha = 1/2$, $\gamma = 1$ and $-\alpha - \beta - 2\gamma(n-1) = -2p$, so that $\beta = -2M + 2p + 3/2$, we require $p > M - \frac{3}{4}$, which gives the result:

$$\int_{-\infty}^{\infty} d\boldsymbol{\lambda} \Delta^2(\boldsymbol{\lambda}^2) \prod_{j=1}^M (1 + \lambda_j^2)^{-2p} = \prod_{j=0}^{M-1} \frac{\Gamma(2+j) \Gamma(\frac{1}{2}+j) \Gamma(-2M+2p+3/2+j)}{\Gamma(-M+2p+j+1)}. \quad (4.17)$$

Therefore, one fermionic resolution of unity in the most general symmetry class D is given by:

$$\hat{I} = \int d\mathbf{H} \hat{\Lambda}(\mathbf{H}) P^{(1)}(\mathbf{H}^2), \quad (4.18)$$

where the normalization constant $C^{(1)}$ is given by:

$$C^{(1)} = \frac{2^{M^2}}{\pi^{M(M-\frac{1}{2})}} \prod_{j=0}^{M-1} \frac{\Gamma(-M+2p+j+1)}{\Gamma(-2M+2p+j+\frac{3}{2})}. \quad (4.19)$$

2. Gaussian normalization

Now we consider the second option for the normalization function, given in Eq. (4.3), where we normalize the distribution by a Gaussian c-number function of the eigenvalues, as often used in random matrix theory. Since this normalization is another even function, the integral over Gaussian operators reduces to a term proportional to the identity operator, as before. We now wish to evaluate the normalization constant, to obtain a resolution of unity. In this case the integral over the radial part is:

$$\int_{-\infty}^{\infty} d\boldsymbol{\lambda} \Delta^2(\boldsymbol{\lambda}^2) P^{(2)}(\boldsymbol{\lambda}^2) = C^{(2)} \int_{-\infty}^{\infty} d\boldsymbol{\lambda} \Delta^2(\boldsymbol{\lambda}^2) e^{-2p \sum_j \lambda_j^2}. \quad (4.20)$$

The integral of Eq. (4.20) is a Selberg type integral related to the Laguerre polynomials³⁸:

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |\Delta(x^2)|^{2\gamma} \prod_{j=1}^n |x_j|^{2\tilde{\alpha}-1} \exp\left(-\frac{x_j^2}{2}\right) d\mathbf{x} = 2^{\tilde{\alpha}n + \gamma n(n-1)} \prod_{j=1}^n \frac{\Gamma(1+j\gamma) \Gamma(\tilde{\alpha} + \gamma(j-1))}{\Gamma(1+\gamma)}.$$

We set $\tilde{\alpha} = 1/2$, $\gamma = 1$, $n = M$ and $x_j^2 = 4p\lambda_j$. Hence we obtain:

$$\int_{-\infty}^{\infty} P^{(2)}(\boldsymbol{\lambda}^2) \Delta^2(\boldsymbol{\lambda}^2) d\boldsymbol{\lambda} = (2p)^{-M(M-\frac{1}{2})} \prod_{j=1}^M \Gamma(1+j) \Gamma\left(j - \frac{1}{2}\right).$$

In this case the resolution of unity is

$$\hat{I} = \int d\mathbf{H} \hat{\Lambda}(\mathbf{H}) P^{(2)}(\mathbf{H}^2), \quad (4.21)$$

where the normalizing constant is given by

$$C^{(2)} = 2^{M^2} \left(\frac{2p}{\pi}\right)^{M(M-1/2)}. \quad (4.22)$$

V. NUMBER-CONSERVING GAUSSIAN OPERATORS

By analogy to the general Gaussian operators, we wish to investigate if there is a similar expression for the fermionic identity operator in terms of the number-conserving Gaussian operators. These operators, $\hat{G}_N(\mathbf{h})$, defined in Eq. (2.15) can be normalized so that $\text{Tr}[\hat{G}_N(\mathbf{h})] = 1$. The normalized number-conserving Gaussian operator $\hat{\Lambda}_N(\mathbf{h})$ is:

$$\hat{\Lambda}_N(\mathbf{h}) = \frac{e^{-\frac{1}{2}\text{Tr}(\mathbf{h})}}{\det(2 \cosh(\mathbf{h}/2))} : \exp \left[\hat{\mathbf{a}}^\dagger \left[e^{\mathbf{h}} - \mathbf{I} \right] \hat{\mathbf{a}} \right] : . \quad (5.1)$$

In section II C, we introduced the variable $\mathbf{u} = e^{\mathbf{h}}$. We notice that if we define the variable \mathbf{u} as $\mathbf{u} = \tilde{\mathbf{n}}^{-T} - \underline{\mathbf{I}}$, we obtain the expression for the normalized number-conserving Gaussian operator defined in terms of the stochastic Green's functions^{12,13} for particles, \mathbf{n} and holes $\tilde{\mathbf{n}} = \underline{\mathbf{I}} - \mathbf{n}$:

$$\hat{\Lambda}_N(\mathbf{h}) = \det[\tilde{\mathbf{n}}] : \exp \left[\hat{\mathbf{a}}^\dagger [\tilde{\mathbf{n}}^{-1} - 2\mathbf{I}]^T \hat{\mathbf{a}} \right] : . \quad (5.2)$$

A. Matrix polar coordinates

We wish to investigate if there is a normalization factor $\mathcal{N}(\mathbf{h})$ that generates a resolution of unity for the normalized number-conserving Gaussian operators, $\hat{\Lambda}_N(\mathbf{h})$ defined in Eq. (5.1), of the form:

$$\int \hat{\Lambda}_N(\mathbf{h}) \mathcal{N}(\mathbf{h}) d\mathbf{h} = \hat{I}. \quad (5.3)$$

Here $d\mathbf{h}$ is an integration measure over the hermitian matrices \mathbf{h} and $\mathcal{N}(\mathbf{h})$ is a normalization function. This normalization function is defined in order to ensure the convergence of the integral. We can consider, for example, the following option for the normalization, in analogy with the previous case:

$$\mathcal{N}(\mathbf{h}) = \frac{C}{\det[\cosh(\mathbf{h}/2)]^p}, \quad (5.4)$$

where C is a normalization constant and we require $p > 1$.

Hermitian matrices can be decomposed in polar coordinates as³⁹ $\mathbf{h} = \mathbf{U}\boldsymbol{\lambda}\mathbf{U}^\dagger$, where \mathbf{U} is an unitary matrix and corresponds to the angular coordinates, while $\boldsymbol{\lambda}$ corresponds to the radial coordinates and is a diagonal matrix, $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_M\}$. The Jacobian of the transformation from the cartesian coordinates \mathbf{h} to polar coordinates $(\boldsymbol{\lambda}, \mathbf{U})$ is given by⁴⁰:

$$d\mathbf{h} = d\mathbf{U} \Delta^2(\boldsymbol{\lambda}) \prod_i d\lambda_i. \quad (5.5)$$

Here $d\mathbf{U}$ is the normalized Haar measure over unitary matrices and $\Delta(\boldsymbol{\lambda})$ is the Vandermonde determinant defined in Eq. (4.5).

B. Evaluation of integrals

Hence, on diagonalizing the Gaussian operator:

$$\hat{\Lambda}_N(\mathbf{h}) = : \exp \left[\hat{\mathbf{b}}^\dagger \left(e^{\boldsymbol{\lambda}} - \mathbf{I} \right) \hat{\mathbf{b}} \right] : \prod_{j=1}^M \frac{e^{-\frac{1}{2}\lambda_j}}{2 \cosh(\lambda_j/2)}, \quad (5.6)$$

where we have defined $\hat{\mathbf{b}} = \mathbf{U}^\dagger \hat{\mathbf{a}}$. Therefore Eq. (5.3) in polar coordinates is:

$$\int \hat{\Lambda}_N(\mathbf{h}) \mathcal{N}(\mathbf{h}) d\mathbf{h} = C \int d\mathbf{U} \Delta^2(\boldsymbol{\lambda}) \prod_{j=1}^M \left[\frac{e^{-\frac{1}{2}\lambda_j} : \exp \left[\hat{b}_j^\dagger (e^{\boldsymbol{\lambda}} - 1)_j \hat{b}_j \right] :}{2 \cosh(\lambda_j/2)^{p+1}} \right] d\lambda_j.$$

We will now focus on the integral over the radial part. Using the expression of the operator term defined in Eq. (4.7), we obtain:

$$\begin{aligned} \int \hat{\Lambda}_N(\mathbf{h}) \mathcal{N}(\mathbf{h}) d\mathbf{h} &= C \int d\mathbf{U} \Delta^2(\boldsymbol{\lambda}) \prod_{j=1}^M \left(1 + \hat{b}_j^\dagger (e^{\lambda_j} - 1) \hat{b}_j \right) \left[\frac{e^{-\lambda_j/2}}{2 \cosh(\lambda_j/2)^{p+1}} \right] d\lambda_j \\ &= C \int d\mathbf{U} \Delta^2(\boldsymbol{\lambda}) \prod_{j=1}^M \left(\frac{e^{-\lambda_j/2}}{\cosh(\lambda_j/2)^{p+1}} + \frac{\hat{b}_j^\dagger \hat{b}_j \sinh(\lambda_j/2)}{\cosh(\lambda_j/2)^{p+1}} \right) d\lambda_j. \end{aligned}$$

Here we notice that the Vandermonde determinant term is not an even function of each eigenvalue. Hence, we do not obtain a resolution of unity following simple parity arguments with an even weight function, as in the previous section. The reason is that when expanding the terms of the Vandermonde determinant, $\prod_{i < j} (\lambda_i - \lambda_j)$, for different values of i and j , we eventually obtain operator terms $\hat{b}_j^\dagger \hat{b}_j$ that depend on even *and* odd functions of λ_j for a fixed value of p .

On the other hand, if $\mathcal{N}(\mathbf{h})$ is not unitarily invariant, and includes terms that when multiplied by the Vandermonde determinant give an even function of λ_i , then we reach a different conclusion. An example of this is if the weight function has the form:

$$\mathcal{N}(\mathbf{h}) = \left[\prod_{1 \leq i < j \leq M} (\lambda_i + \lambda_j)^2 \right] e^{-p \sum_i \lambda_i^2} \quad (5.7)$$

In this case, it is clear that:

$$\Delta^2(\boldsymbol{\lambda}) \mathcal{N}(\mathbf{h}) = \left[\prod_{1 \leq i < j \leq M} (\lambda_i^2 - \lambda_j^2)^2 \right] e^{-p \sum_i \lambda_i^2}. \quad (5.8)$$

With such a weight, the integral corresponding to the operator terms in Eq. (5.3) vanishes, because of the parity of the functions: the function $\sinh(\lambda_j/2)$ is odd while the other are even functions of λ_j . In this case we can obtain a resolution of unity just as in the nonstandard symmetry case. Therefore, for the number-conserving Gaussian operators, we can obtain a resolution of unity if the normalization factor cancels the parity violation of the Vandermonde terms. Thus, we have shown that an expansion of the identity operator is also possible with the number-conserving Gaussians. As this is not a simple trace or determinant, we cannot easily express this in terms of the original coordinates \mathbf{h} . These arguments obviously do not exclude other routes to obtaining a resolution of unity with these operators.

VI. SUMMARY

We have studied a subset of the general fermionic Gaussian operators, the hermitian positive definite fermionic Gaussian operators, which belong to a non-standard symmetry class of random matrices. We have proved that there are simple resolutions of unity for these operators, for any of the non-standard symmetry classes, and for any integrable even distribution of eigenvalues. This resolution of unity is defined for the entire Hilbert space, not for a coset or subset space as in the case of the fermionic coherent states.

Our proof is based on considering the hermitian Gaussian operators as an operator basis depending on a continuous hermitian matrix. We use polar coordinates for skew-symmetric matrices, which lead to an expression for the resolution of unity in terms of integrals over eigenvalues. Our result appears similar to related expressions in random matrix theory. In order to obtain this unitarily-invariant expression for the resolution of the unity, the normalizing factor must be an even function of the eigenvalues. This suggests that distributions of random matrices in the nonstandard symmetry classes, if used to average over canonical density matrices, should generally have other types of distribution. Otherwise the physical behaviour will simply correspond to a unit density matrix. While this is not impossible, it is unlikely to be generic. In the case of the symmetry class D, we give the values of the constants corresponding to the angular and radial integrals, with two different options for the normalizing factor.

These resolutions of unity for the Gaussian operators can be used to derive mathematical identities for other physical applications. As an example, our results can be used to construct a positive fermionic distribution or fermionic Q-function. This will be carried out elsewhere.

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Appendix A: integral over the angular coordinates

In this Appendix we evaluate the integral over the angular coordinates \mathbf{U} , whose expression is given in Eq. (4.13), for the non-standard symmetry class D.

We wish to evaluate the constant $C^{\mathbf{U}} = \int (\mathbf{U}^\dagger d\mathbf{U})$. This is the integral over the angular variables \mathbf{U} . It is the angular part of the Jacobian of the matrix transformation from Cartesian coordinates $\mathbf{H} = i\mathbf{X}$ to polar coordinates $(\mathbf{U}, \boldsymbol{\lambda})$:

$$\int d\mathbf{H} P(\mathbf{H}^2) = \int (\mathbf{U}^\dagger d\mathbf{U}) \int d\boldsymbol{\lambda} \Delta^2(\boldsymbol{\lambda}^2) \prod_{j=1}^M P(\lambda_j^2). \quad (\text{A1})$$

The value of the integral over the angular variables \mathbf{U} is given by the ratio of an integral over cartesian coordinates \mathbf{H} , to an integral over radial coordinates $\boldsymbol{\lambda}$:

$$C^{\mathbf{U}} = \frac{\int d\mathbf{H} P(\mathbf{H}^2)}{\int d\boldsymbol{\lambda} \Delta^2(\boldsymbol{\lambda}^2) P(\lambda_j^2)}. \quad (\text{A2})$$

Since the constant $C^{\mathbf{U}}$ is evaluated as a ratio, we will perform the calculations using the Gaussian form of the normalization function. This gives simple integrals in Cartesian matrix coordinates. For these calculations, we therefore use:

$$P(\mathbf{H}^2) = \exp[-p \text{Tr}[\mathbf{H}^2]]. \quad (\text{A3})$$

Consequently, we now have to evaluate:

$$C^{\mathbf{U}} = \frac{\int d\mathbf{H} \exp[-p \text{Tr}[\mathbf{H}^2]]}{\int d\boldsymbol{\lambda} \Delta^2(\boldsymbol{\lambda}^2) e^{-2p \sum_j \lambda_j^2}}. \quad (\text{A4})$$

Radial integrals

The radial integral over the eigenvalues λ_j in Eq. (A4) is a Selberg type integral related to the Laguerre polynomials³⁸:

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |\Delta(x^2)|^{2\gamma} \prod_{j=1}^n |x_j|^{2\tilde{\alpha}-1} \exp(-x_j^2/2) d\mathbf{x} = 2^{\tilde{\alpha}n + \gamma n(n-1)} \prod_{j=1}^n \frac{\Gamma(1+j\gamma) \Gamma(\tilde{\alpha} + \gamma(j-1))}{\Gamma(1+\gamma)} \quad (\text{A5})$$

In our case, we consider $\gamma = 1$, $\tilde{\alpha} = \frac{1}{2}$, so that the radial integration gives:

$$\int_{-\infty}^{\infty} d\boldsymbol{\lambda} \Delta^2(\boldsymbol{\lambda}^2) e^{-2p \sum_j \lambda_j^2} = (2p)^{-M(M-1)} \prod_{j=1}^M \Gamma(1+j) \Gamma\left(j - \frac{1}{2}\right). \quad (\text{A6})$$

Cartesian integrals

The next step is evaluate the integral over the $2M \times 2M$ hermitian matrix \mathbf{H} , defined as:

$$\mathbf{H} = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}, \quad (\text{A7})$$

with $h_{\alpha\beta} = h_{\beta\alpha}^*$ and $\Delta_{\alpha\beta} = -\Delta_{\beta\alpha}$. We consider the integral over the matrices h and Δ so that:

$$\int d\mathbf{H} \exp[-\text{Tr}(p\mathbf{H}^2)] = \int \prod_{i=1}^M dh_{ii} \prod_{i<j} dh_{ij}^x dh_{ij}^y d\Delta_{ij}^x d\Delta_{ij}^y \exp[-p\text{Tr}(\mathbf{H}^2)]. \quad (\text{A8})$$

The trace of \mathbf{H}^2 can be written as:

$$\begin{aligned} \text{Tr}[\mathbf{H}^2] &= \sum_{ij} |H_{ij}|^2 \\ &= 2 \sum_i h_{ii}^2 + 4 \sum_{i<j} \left[(h_{ij}^x)^2 + (h_{ij}^y)^2 + (\Delta_{ij}^x)^2 + (\Delta_{ij}^y)^2 \right]. \end{aligned} \quad (\text{A9})$$

Hence, we can use the result that:

$$\int_{-\infty}^{\infty} dh e^{-2ph^2} = \sqrt{\frac{\pi}{2p}}, \quad (\text{A10})$$

to obtain the overall Cartesian integral of:

$$\int d\mathbf{H} \exp[-p\text{Tr}(\mathbf{H}^2)] = \left(\sqrt{\frac{\pi}{2p}} \right)^{M(2M-1)} 2^{-M(M-1)}, \quad (\text{A11})$$

where we have integrated over both the diagonal and the off-diagonal terms.

Normalization constant

Using the results of Eq. (A6) and Eq. (A11) we obtain:

$$C^{\text{U}} = \pi^{M(M-\frac{1}{2})} 2^{-M(M-1)} \prod_{j=0}^{M-1} \frac{1}{\Gamma(2+j) \Gamma(j+\frac{1}{2})}. \quad (\text{A12})$$

REFERENCES

- ¹R. Balian and E. Brezin, *Il Nuovo Cimento B* **64**, 37 (1969).
- ²A. Altland and M. R. Zirnbauer, *Phys. Rev. B* **55**, 1142 (1997).
- ³F. J. Dyson, *J. Math. Phys.* **3**, 1199 (1962).
- ⁴F. J. Dyson, *J. Math. Phys.* **3**, 140 (1962).
- ⁵F. J. Dyson, *Commun. Math. Phys.* **19**, 1235 (1970).
- ⁶H. Weyl, *The classical groups: Their invariants and representations* (Princeton University Press, Princeton, N.J., 1939).
- ⁷A. M. Perelomov, *Generalized coherent states and their applications*, Texts and monographs in physics (Springer, Berlin, 1986).

- ⁸W.-M. Zhang, D. H. Feng, and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
- ⁹R. Gilmore and D. H. Feng, Prog. Part. Nucl. Phys. **9**, 479 (1983).
- ¹⁰P. J. Forrester, *Log-gases and random matrices* (Princeton University Press, Princeton, 2010).
- ¹¹M. Caselle and U. Magnea, Phys. Rep. **394**, 41 (2004).
- ¹²J. F. Corney and P. D. Drummond, J. Phys. A **39**, 269 (2006).
- ¹³J. F. Corney and P. D. Drummond, Phys. Rev. B **73**, 125112 (2006).
- ¹⁴J. F. Corney and P. D. Drummond, Phys. Rev. Lett. **93**, 260401 (2004).
- ¹⁵T. Aimi and M. Imada, J. Phys. Soc. Jpn. **76**, 084709 (2007).
- ¹⁶L. E. C. Rosales-Zárate and P. D. Drummond, Phys. Rev. A **84**, 042114 (2011).
- ¹⁷R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
- ¹⁸J. R. Klauder and B.-S. Skagerstam, *Coherent states: Applications in physics and mathematical physics* (World Scientific, Singapore, 1985).
- ¹⁹J. R. Klauder, Ann. Phys. **11**, 123 (1960).
- ²⁰E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
- ²¹K. Husimi, Proc. Phys. Math. Soc. Jpn **22**, 264 (1940).
- ²²R. Gilmore, Ann. Phys. **74**, 391 (1972).
- ²³R. Gilmore, Rev. Mex. Fis. **23**, 143 (1974).
- ²⁴A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972).
- ²⁵A. M. Perelomov, Sov. Phys. Usp. **20**, 703 (1977).
- ²⁶J. P. Blaizot and H. Orland, Phys. Rev. C **24**, 1740 (1981).
- ²⁷J.-P. Blaizot and G. Ripka, *Quantum theory of finite systems* (MIT Press, Cambridge, Mass., 1986).
- ²⁸D. J. Rowe and A. G. Ryman, Phys. Rev. Lett. **45**, 406 (1980).
- ²⁹D. J. Rowe, T. Song, and H. Chen, Phys. Rev. C **44**, R598 (1991).
- ³⁰J.-P. Gazeau, *Coherent States in Quantum Physics* (Wiley-VCH, Weinheim, 2009).
- ³¹K. E. Cahill and R. J. Glauber, Phys. Rev. A **59**, 1538 (1999).
- ³²Y. Ohnuki and T. Kashiwa, Prog. Theor. Phys. **60**, 548 (1978).
- ³³J. W. Negele and H. Orland, *Quantum many-particle systems* (Perseus Books, Reading, MA., 1998).
- ³⁴F. A. Berezin, *The method of second quantization* (Academic Press, New York, 1966).
- ³⁵H.-Y. Fan, Ann. Phys. **322**, 886 (2007).
- ³⁶E. P. Wigner, Proc. Cambridge Phil. Soc. **47**, 790 (1951).

³⁷E. P. Wigner, *Ann. Math.* **67**, 325 (1958).

³⁸M. L. Mehta, *Random matrices*, 3rd ed. (Academic Press, Boston, 2004).

³⁹L.-K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains* (American Mathematical Society, Providence, Rhode Island, 1963).

⁴⁰C. Itzykson and J.-B. Zuber, *J. Math. Phys.* **21**, 411 (1980).